

The totally nonnegative part of the finite Toda lattice via a reducible rational curve

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Abstract

A totally nonnegative matrix is a real-valued matrix whose minors are all nonnegative. In this paper, we concern with the totally nonnegative structure of the finite Toda lattice, a classical integrable system, which is expressed as a differential equation of square matrices. The Toda flow naturally translates into a (multiplicative) linear flow on the (generalized) Jacobi variety associated with some reducible rational curve X . This correspondence provides an algebro-geometric characterization of the totally positive part of the Toda equation. We prove that the totally nonnegative part of the finite Toda lattice is isomorphic to a connected component of $\text{Jac}(X)_{\mathbb{R}}$, the real part of the generalized Jacobi variety $\text{Jac}(X)$, as semi-algebraic varieties.

1 Introduction

A real matrix is said to be *totally positive* (*resp. totally nonnegative*) if all its minors are positive (*resp. nonnegative*). In this article, we study the *totally nonnegative part* of the phase space of the *finite Toda lattice*, that is one of the most basic classical integrable system. Since the Toda lattice was originally discovered as a physical system, it would be natural to expect that the totally nonnegative part possesses all the essential structures of the system, although it is “mathematically natural” to consider the phase space of complex numbers. We give an algebro-geometric characterization of the totally nonnegative part of the finite Toda lattice via the Krichever construction associated with a singular algebraic curve.

The aim

The methods to construct solutions of the finite Toda lattice were mainly developed in 1970’s by many authors. It is well-known that there exist quite a few approaches to give explicit solutions such as the *inverse scattering method* [1], the method of the *QR decomposition* [4, 8], the method of *Wronskian matrix solutions*, the *continued fraction expansion method* [7], *etc.*

In this study, we use so-called *Krichever construction* associated with some singular curve [5, 6]. In 2002, Krichever and Vaninsky [6, 14] pointed out the relation between the finite Toda lattice and

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some kind of reducible curve, and found the fact that the Baker-Akhiezer function associated with that curve gives rational solutions of the finite Toda lattice. After that, Sklyanin [10] obtained more explicit expressions of these rational solutions in terms of the *dual variable* by making more detailed analysis on the reducible curve.

Inspired by their works [6, 10], we give an geometric interpretation of the totally non-negative part of the finite Toda lattice by re-interpreting their construction as a *generalized Jacobi inversion problem*. We are also inspired by the paper [2] in 2014 by Kodama and Williams, who have investigated the combinatorial structure of the totally nonnegative part of the system. (In [2], they have also studied the full Kostant-Toda hierarchy, which is a generalization of the Toda lattice. It should be an interesting problem to give a similar characterization of the totally nonnegative part of it. Also see [3] for a detailed survey of the topological properties of the finite Toda lattice.)

The outline of the result of this paper is as follows: Let $L \in M_N(\mathbb{C})$ be a complex matrix of size N . Its characteristic polynomial $\det(\lambda E - L) = \lambda^N - H_1(L)\lambda^{N-1} + H_2(L)\lambda^{N-2} - \dots + (-1)^N H_N(L)$ defines the algebraic map

$$\begin{array}{ccccc} M_N(\mathbb{C}) & \xrightarrow{H} & \mathbb{C}^N & \xleftarrow{\gamma} & \mathbb{C}^N/\mathfrak{S}_N \\ L & \mapsto & (H_1(L), \dots, H_N(L)) & & \\ & & (e_1(\Lambda), \dots, e_N(\Lambda)) & \leftarrow & \Lambda = \{\lambda_1, \dots, \lambda_N\}, \end{array}$$

where $e_i(\Lambda)$ is the i -th symmetric function in $\lambda_1, \dots, \lambda_N$. Let Γ (Eq. (1)) be the phase space of the finite Toda lattice (Eq. (2)). The subset $\mathcal{T}_\Lambda := H^{-1}(\gamma(\Lambda)) \cap \Gamma$ is called an *isospectral set*. The Toda equation determines a time-independent flow (so-called the *Toda flow*) on each isospectral set \mathcal{T}_Λ . For any Λ , the isospectral set \mathcal{T}_Λ is connected and of complex dimension $N - 1$ as a subvariety of $M_N(\mathbb{C})$.

We consider the set $\mathcal{T}_\Lambda^\geq := \mathcal{T}_\Lambda \cap \{\text{totally non-negative matrices}\}$, the *totally non-negative part* of \mathcal{T}_Λ . It is known that, if \mathcal{T}_Λ^\geq is non-empty, the numbers $\lambda_1, \dots, \lambda_N \in \Lambda$ must be distinct, real and positive. (Theorem 3.2).

The following is the main theorem of this paper:

Theorem 1.1 (Theorem 3.9). Assume $\mathcal{T}_\Lambda^\geq \neq \emptyset$. Then, the image of \mathcal{T}_Λ^\geq by the *linearization map* $\Phi : \mathcal{T}_\Lambda \rightarrow \text{Jac}(X)$ (Eq. (16)) is described as follows:

$$\Phi(\mathcal{T}_\Lambda^\geq) = \{[F_1 : F_2 : \dots : F_N] \in \text{Jac}(X) \mid (-1)^i F_i > 0 \text{ for all } i\},$$

where X is a reducible curve defined in §2.2, and $\text{Jac}(X) \simeq (\mathbb{C}^\times)^N / \mathbb{C}^\times$ is the (generalized) Jacobi variety associated with X .

Especially, the image $\Phi(\mathcal{T}_\Lambda^\geq)$ is a positive cone of (real) dimension $N - 1$. The Abel-Jacobi map Φ induces the isomorphism

$$\Phi|_{\mathcal{T}_\Lambda^\geq} : \mathcal{T}_\Lambda^\geq \rightarrow \text{Jac}(X)_\mathbb{R}^0$$

of semi-algebraic varieties, where $\text{Jac}(X)_\mathbb{R}^0$ is the connected component of $\text{Jac}(X)_\mathbb{R}$ (§3.3) which contains the identity element $[1 : 1 : \dots : 1]$.

This result provides several properties of the Toda flow. For example, the totally non-negative part \mathcal{T}_Λ^\geq is connected and closed under the Toda flow. Any orbit of a totally non-negative matrix never contain a blowup point.

Contents of the paper

The contents of the paper is as follows:

In Section 2, we demonstrate the algebro-geometric construction of determinant solutions of the Toda lattice by solving the (generalized) Jacobi inversion problem associated with a singular spectral

curve. The singular curve which we use here is the same thing as in the papers [6, 14]. Although the conclusion (Theorem 2.11) of this section is not new at all, our method provides more suitable geometric interpretations of some datum related to the curve. (For example, the “gluing condition” in [6, §3] can be understood as an element of $\text{Jac}(X)$ in our context).

In Section 3, we give a proof of the main theorem (Theorem 3.9). Our proof (§3.2) essentially depends on the explicit form of the determinant solution.

Remark: two meanings of “totally non-negativity”.

When referring to the “totally non-negativity” of the Lax equation $\frac{d}{dt}L = [L, L_-]$, we should be careful not to confuse the following two meanings:

1. the totally non-negativity of the *companion matrix* of L .
2. the totally non-negativity of L itself.

Many researchers maybe familiar with the first one, that has been well-studied in researches on blowup solutions of the Toda equation. (The “TNN part” of Kodama-Williams belongs to this.) On the other hand, among researches on discrete integrable systems and their tropicalizations, the totally non-negativity in the second meaning is a main object to be studied. It is straightforward to check that the totally non-negativity in the second meaning implies the first one.

In this study, we always refer to the totally non-negativity in the second meaning.

2 The solution of the finite Toda lattice in terms of singular theta functions

2.1 The finite Toda lattice in Lax formalism

Let N be a positive integer. Put $G = GL_N(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$. Define the subgroups $B^+, U^- \in G$ by

$$B^+ := \{x \in G \mid \text{upper triangular}\}, \quad U^- := \{x \in G \mid \text{lower triangular with diagonals 1}\}$$

and the subalgebras $\mathfrak{g}_\pm \subset \mathfrak{g}$ by

$$\mathfrak{g}_+ := \{x \in \mathfrak{g} \mid \text{upper triangular}\} = \text{Lie}(B^+), \quad \mathfrak{g}_- := \{x \in \mathfrak{g} \mid \text{strictly lower triangular}\} = \text{Lie}(U^-).$$

It is well-known that the product map $B^+ \times U^- \rightarrow G; (b, u) \mapsto b \cdot u$ is an open embedding so that we have $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. For $X \in \mathfrak{g}$, denote by $X = X_+ + X_-$ ($X_+ \in \mathfrak{g}^+$, $X_- \in \mathfrak{g}^-$) the decomposition of X along the direct sum decomposition.

Let $\Gamma \subset \mathfrak{g}$ be a subset defined by

$$\Gamma := \left\{ \begin{pmatrix} a_1 & 1 & & & \\ b_1 & a_2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ & & & b_{N-1} & a_N \end{pmatrix} \in \mathfrak{g} ; b_1 b_2 \cdots b_{N-1} \neq 0 \right\}. \quad (1)$$

In this section, we consider the *finite Toda lattice* [12, 13]:

$$\frac{d}{dt}L = [L, L_-], \quad L = L(t) \in \Gamma. \quad (2)$$

Fix an initial value $L_0 = L(0) \in \Gamma$ of the system (2). For a real number t , $\exp(tL_0) = a(t)^{-1}b(t) \in B^+ \cdot U^-$ be the LU decomposition.

Lemma 2.1 ([11]). $L(t) := a(t)L_0a(t)^{-1} = b(t)L_0b(t)^{-1}$ solves the equation (2). \square

2.1.1 The Toda flow on the spectrum set

From the Lax form (2), the characteristic polynomial

$$f(\lambda) := (-1)^N \det(L(t) - \lambda \cdot E) = (-1)^N \det(L_0 - \lambda \cdot E) \quad (3)$$

is independent of t . Since we are interested in the totally non-negative structure, we restrict ourselves to the case when all the roots of $f(\lambda)$ are simple (see Theorem 3.2). We denote these roots by $\lambda_1, \dots, \lambda_N \in \mathbb{C}$. Let C be the finite set $C := \{\lambda_1, \dots, \lambda_N\} \subset \mathbb{C}$, the spectrum set of $L(t)$. The ring $\mathcal{O} := \mathbb{C}[\lambda]/(f(\lambda))$ is regarded as the ring of algebraic functions over C . By the Chinese remainder theorem, \mathcal{O} admits the ring isomorphism

$$\mathcal{O} \xrightarrow{\sim} \mathbb{C}^N; \quad \varphi \mapsto (\varphi(\lambda_1), \dots, \varphi(\lambda_N)). \quad (4)$$

For each $t \in \mathbb{R}$, consider a column vector $\mathbf{v}(t) := (v_1(t), \dots, v_N(t))^T$, $(v_i(t) \in \mathcal{O})$ which satisfies the linear equation

$$(L(t) - \lambda E)\mathbf{v}(t) \equiv \mathbf{0} \pmod{f(\lambda)}. \quad (5)$$

The cofactor expansion of $(L(t) - \lambda E)$ along the N -th row yields the column vector

$$\mathbf{v}_-(t) = ((-1)^{N+j} \Delta_{N,j}(t; \lambda))_{j=1}^N, \quad \text{where } \Delta_{i,j} \text{ is a } (i, j)\text{-minor of } (L(t) - \lambda E).$$

As a polynomial in λ , the i -th component of $\mathbf{v}_-(t)$ is expressed as $\lambda^{i-1} + O(\lambda^{i-2})$. On the other hand, the expansion along the first row yields the column vector

$$\mathbf{v}_+(t) = ((-1)^{1+j} \Delta_{1,j}(t; \lambda))_{j=1}^N$$

whose i -th components is of the form $(b_1 b_2 \cdots b_{i-1}) \lambda^{N-i} + O(\lambda^{N-i-1})$. Since the matrix $(L(t) - \lambda E)$ is always expressed as $P \cdot \text{diag}(1, \dots, 1, f(\lambda)) \cdot Q$, (P, Q are invertible matrix over $\mathbb{C}[\lambda]$), the two vectors $\mathbf{v}_-(t)$ and $\mathbf{v}_+(t)$ are in proportion:

$$F(t) \cdot \mathbf{v}_-(t) = \mathbf{v}_+(t), \quad \exists! F(t) \in \mathcal{O}^\times. \quad (6)$$

Comparing the first components on the both sides, we have $F(t) = \Delta_{1,1}(t; \lambda)$. Therefore, $F(t)$ is nothing but the *chop integral* introduced in [2, 3].

The time dependence of $F(t)$ is calculated as follows. From $L(t) = a(t)L_0 a(t)^{-1} = b(t)L_0 b(t)^{-1}$, we have two equations

$$L(t)\{a(t)\mathbf{v}(0)\} = \lambda\{a(t)\mathbf{v}(0)\}, \quad L(t)\{b(t)\mathbf{v}(0)\} = \lambda\{b(t)\mathbf{v}(0)\}.$$

Therefore, all the vectors $\mathbf{v}_\pm(t)$, $a(t)\mathbf{v}_\pm(0)$, $b(t)\mathbf{v}_\pm(0)$ are in proportion. From the facts $a(t) \in U^-$ and $b(t) \in B^+$, we conclude

1. $\mathbf{v}_-(t) = a(t)\mathbf{v}_-(0)$ and
2. $\mathbf{v}_+(t) = k(t)b(t)\mathbf{v}_+(0)$ for some non-zero complex number $k(t) \in \mathbb{C}^\times$.

Combining these equations and (6), we obtain $F(t) \cdot a(t)\mathbf{v}_-(0) = k(t)b(t)\mathbf{v}_+(0)$. Therefore, $F(t) \cdot \mathbf{v}_-(0) = k(t)a(t)^{-1}b(t)\mathbf{v}_+(0) = k(t)\exp(tL_0)\mathbf{v}_+(0) = k(t)\exp(t\lambda)\mathbf{v}_+(0)$. This yields

$$F(t) = k(t)\exp(t\lambda) \cdot F(0), \quad F(t) \in \mathcal{O}^\times \quad (7)$$

or in other words,

$$\overline{F}(t) = \exp(t\lambda) \cdot \overline{F}(0), \quad \overline{F}(t) \in \mathcal{O}^\times / \mathbb{C}^\times, \quad (8)$$

where $\overline{F}(t) = F(t) \bmod \mathbb{C}^\times$.

2.2 The generalized Jacobi variety of the singular spectral curve.

Denote $\mathbb{P} := \mathbb{P}^1(\mathbb{C})$. Let X_- and X_+ be a pair of copies of \mathbb{P} with coordinate variables x and y , respectively. Define the reduced, reducible and nodal curve X by gluing X_- and X_+ along $\{x = \lambda_i\} \in X_-$ and $\{y = \lambda_i\} \in X_+$, ($i = 1, 2, \dots, n$) transversally. Let $P_1, \dots, P_N \in X$ be the singular points (nodes) of X and $\infty_+, \infty_- \in X$ be the points at infinity on X_- and X_+ , respectively¹. Later we will assign this singular curve X with spectral datum of the finite Toda lattice.

2.2.1 Definition of the generalized Abel-Jacobi map

Let $H(X)$ be the group

$$H(X) := \left\{ (f, g) \left| \begin{array}{l} \text{(i) } f \text{ (resp. } g) \text{ is a rational function over } X_- \text{ (resp. } X_+), \\ \text{(ii) } f \text{ and } g \text{ have no poles and zeros at } P_1, \dots, P_N, \\ \text{(iii) } f(P_n) = g(P_n), \quad n = 1, \dots, N \end{array} \right. \right\},$$

where the product is given by $(f, g) \cdot (f', g') := (ff', gg')$.

Let $\text{Div}(X) := \bigoplus_{p \in X \setminus \{P_1, \dots, P_N\}} \mathbb{Z} \cdot p$ be the divisor group of X and $\text{Div}^d(X)$ be the subset of divisors of degree $d \in \mathbb{Z}$. Further, define the subset $\text{Div}^{a,b}(X) \subset \text{Div}^{a+b}(X)$ by

$$\text{Div}^{a,b}(X) := \{D \in \text{Div}^{a+b}(X) \mid \deg(D \cap X_-) = a, \deg(D \cap X_+) = b\}.$$

It follows that $\text{Div}^d(X) = \coprod_{a+b=d} \text{Div}^{a,b}(X)$.

For a pair of non-zero rational functions $v = (f, g)$, (f (resp. g) is a rational function over X_- (resp. X_+)), define the divisor $(v) \in \text{Div}^{0,0}(X)$ by $(v) := (\text{the zeros of } v) - (\text{the poles of } v) = (v)_0 - (v)_\infty$.

For a divisor $D \in \text{Div}^{0,0}(X)$, there exists a pair of rational functions (f, g) whose divisor is D . This determines the homomorphism

$$\text{Div}^{0,0}(X) \rightarrow \mathcal{O}^\times / \mathbb{C}^\times \simeq (\mathbb{C}^\times)^N / \mathbb{C}^\times; \quad D \mapsto \left[\frac{g(\lambda_1)}{f(\lambda_1)} : \dots : \frac{g(\lambda_N)}{f(\lambda_N)} \right]. \quad (9)$$

Define the *Picard group* $\text{Pic}^{a,b}(X)$ of X of degree (a, b) by

$$\text{Pic}^{a,b}(X) := \text{Div}^{a,b}(X) / \sim, \quad D_1 \sim D_2 \iff D_1 - D_2 = (v), \quad \exists v \in H(X).$$

The homomorphism (9) induces the map $J : \text{Pic}^{0,0}(X) \rightarrow \mathcal{O}^\times / \mathbb{C}^\times$.

Definition 2.2. We define the (*generalized*) *Jacobi variety* of X as $\text{Jac}(X) := \mathcal{O}^\times / \mathbb{C}^\times$. The map $J : \text{Pic}^{0,0}(X) \rightarrow \text{Jac}(X)$ is called the (*generalized*) *Abel-Jacobi map*.

Proposition 2.3. J is a group isomorphism.

Proof. Assume $J(D) = [1 : \dots : 1]$. Then, there exists some $v = (f, g) \in H(X)$ such that $D = (v)$, which implies $D \equiv 0$. Therefore, J is injective. On the other hand, it is straightforward to prove J to be surjective because there must exist a polynomial $h(\lambda)$ such that $[h(\lambda_1) : \dots : h(\lambda_N)] = F$ for any $F = [F_1 : \dots : F_N] \in \mathcal{O}^\times / \mathbb{C}^\times$. \square

2.2.2 General divisors

Consider the subset

$$S_{a,b} := \{p_1 + \dots + p_a + q_1 + \dots + q_b \in \text{Div}^{a,b}(X) \mid p_i \in X_-, q_i \in X_+\}, \quad (a, b \in \mathbb{Z}_{\geq 0})$$

¹The affine part $X \setminus \{\infty_\pm\}$ is isomorphic to $\text{Spec} [\mathbb{C}[\lambda, \mu] / (\mu^2 - \prod_i (\lambda - \lambda_i)^2)]$.

of formal sums of $(a + b)$ points. We demonstrate the analog of the classical *Jacobi inversion problem*, that asks whether the map

$$\phi_{a,b} : S_{a,b} \rightarrow \text{Jac}(X); \quad D \mapsto J(D - a \cdot \infty_- - b \cdot \infty_+)$$

admits an inverse map.

Proposition 2.4. If $a + b = N - 1$, the map $\phi_{a,b}$ is surjective. Moreover, for an element $A = [z_1 : \cdots : z_N]$ in a generic position of the Jacobi variety $\text{Jac}(X)$, the inverse image $\phi_{a,b}^{-1}(A)$ consists of one element.

Proof. The existence of the inverse image of $[z_1 : \cdots : z_N] \in \text{Jac}(X)$ is equivalent to the existence of the pair of polynomials f, g with $\frac{g(\lambda_n)}{f(\lambda_n)} = z_n$, $\deg f = a$ and $\deg g = b$. Since the total number of coefficients of polynomials f, g is $a + b + 2 = N + 1$, we can conclude the claim by counting the number of variables. \square

By this proposition, if $a + b = N - 1$, we have the relation

$$D \sim D', \quad D' \in S_{a,b}, \quad \Longleftrightarrow \quad \phi_{a,b}(D) = \phi_{a,b}(D') \quad (\text{if } D \text{ is in general position}) \quad D = D'. \quad (10)$$

In general, an element $D \in S_{a,b}$ with the property (10) is called a *general divisor*.

2.2.3 Theta functions

For distinct complex numbers $\lambda_1, \dots, \lambda_N \in \mathbb{C}$, we define the multivalued functions $\theta_k(\cdot; \lambda_1, \dots, \lambda_N) : (\mathbb{C}^\times)^N \rightarrow \mathbb{C}$ for $k = 0, 1, \dots, N$ by the following formula:

$$\theta_k(Z_1, \dots, Z_N; \lambda_1, \dots, \lambda_N) := \frac{1}{\sqrt{Z_1 \cdots Z_N}} \det \left(\begin{array}{cccc|cccc} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^{N-k-1} & Z_1 & Z_1 \lambda_1 & \cdots & Z_1 \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^{N-k-1} & Z_2 & Z_2 \lambda_2 & \cdots & Z_2 \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \lambda_N & \lambda_N^2 & \lambda_N^{N-k-1} & Z_N & Z_N \lambda_N & \cdots & Z_N \lambda_N^{k-1} \end{array} \right). \quad (11)$$

We call θ_k the *degenerated theta function*.

For the reducible curve X defined in previous sections, we refer to the function $\theta_k(Z) = \theta_k(Z; \lambda_1, \dots, \lambda_k)$ as the *theta function associated with X* . The following lemma provides the reason why it can be called the “theta function”.

Lemma 2.5. For $[z_1 : \cdots : z_N] \in \text{Jac}(X)$, we have

$$\theta_k(z_1, \dots, z_N) = 0 \quad \Longleftrightarrow \quad \exists D \in S_{k-1, N-k-1}, \quad \phi_{k-1, N-k-1}(D) = [z_1 : \cdots : z_N].$$

The condition on the right hand side does not depend on the choice of the lift $(z_1, \dots, z_N) \in (\mathbb{C}^\times)^N$. \square

For elements $\mathbf{A} = (A_1, \dots, A_N)$ and $\mathbf{B} = (B_1, \dots, B_N)$ of $(\mathbb{C}^\times)^N$, define $\mathbf{AB} = (A_1 B_1, \dots, A_N B_N)$ and $\frac{\mathbf{A}}{\mathbf{B}} = (\frac{A_1}{B_1}, \dots, \frac{A_N}{B_N})$. Let $\varphi : X \setminus \{P_1, \dots, P_N\} \rightarrow (\mathbb{C}^\times)^N$ be the continuous map

$$\varphi(p) := \begin{cases} (\lambda_1 - \lambda(p), \dots, \lambda_N - \lambda(p))^{-1}, & p \in X_- \\ (\lambda_1 - \lambda(p), \dots, \lambda_N - \lambda(p)), & p \in X_+ \end{cases}.$$

For a fixed element $\mathbf{Z} \in (\mathbb{C}^\times)^N$, define the multivalued function $\Theta_k(\cdot; \mathbf{Z}) : X \setminus \{P_1, \dots, P_N\} \rightarrow \mathbb{C}$ by

$$\Theta_k(p; \mathbf{Z}) := \begin{cases} \theta_{k-1}(\mathbf{Z}/\varphi(p)), & (p \in X_-) \\ \theta_k(\mathbf{Z}/\varphi(p)), & (p \in X_+) \end{cases}.$$

$\Theta_k(p; \mathbf{Z})$ ramifies around $p = P_i$ (the ramification number = 2) and around ∞_\pm (the ramification number = 2 if N is odd, and = 1 if N is even).

Proposition 2.6 (Analog of Riemann's vanishing theorem). For $1 \leq k \leq N$ and $\mathbf{Z} = (Z_1, \dots, Z_N) \in (\mathbb{C}^\times)^N$, fix a divisor $D \in S_{k-1, N-k}$ with $\phi_{k-1, N-k}(D) = [Z_1 : \dots : Z_N]$. Then, the following (i–ii) hold:
(i) $\Theta_k(\cdot; \mathbf{Z})$ is identically 0 if D is non-general.
(ii) $\Theta_k(\cdot; \mathbf{Z})$ has zeros only at $D \in S_{k-1, N-k}$ if D is general. \square

Corollary 2.7. Suppose $D_1 \in S_{k-1, N-k}$ and $D_2 \in S_{l-1, N-l}$ to be general divisors. Let $\mathbf{Z}, \mathbf{Y} \in (\mathbb{C}^\times)^N$ be the lifts of

$$\phi_{k-1, N-k}(D_1), \phi_{l-1, N-l}(D_2) \in \text{Jac}(X) \simeq (\mathbb{C}^\times)^N / \mathbb{C}^\times.$$

Then, the ratio $\frac{\Theta_k(p; \mathbf{Z})}{\Theta_l(p; \mathbf{Y})}$ is a single-valued function over $X \setminus \{P_1, \dots, P_N\}$, whose divisor is $D_1 - D_2 + (k-l)(\infty_+ - \infty_-)$. \square

2.3 The tau functions for the finite Toda lattice

It is classically known that the Toda equation admits a family of algebraic solutions which is expressed as a ratio of determinants. We demonstrate to derive them in terms of theta functions associated with X .

Let $\mathbf{v}_\pm(t)$ be the vector-valued polynomial function in λ defined in §2.1.1. We assign these vectors $\mathbf{v}_\pm(t)$ with a vector-valued function over X in such a way that (i) by replacing λ 's in each component of $\mathbf{v}_-(t)$ with x and (ii) by replacing λ 's in each component of $\mathbf{v}_+(t)$ with y . In this view point, the equation $F(t) \cdot \mathbf{v}_-(t) = \mathbf{v}_+(t)$ (Eq. (6)) represents the perversity of two vector-valued functions $\mathbf{v}_-(t)$, $\mathbf{v}_+(t)$ at the intersection $X_- \cap X_+$.

We regard the pair $\mathbf{v}(t) = (\mathbf{v}_-(t), \mathbf{v}_+(t))$ as one vector-valued function over $X \setminus \{P_1, \dots, P_N\}$. Denote the k -th component of $\mathbf{v}(t)$ by v_k . From (6) and (8), we have

$$\phi_{k-1, N-k}(D_k(t)) = \overline{F}(t) = [\exp(t\lambda_1) : \dots : \exp(t\lambda_N)] \in \text{Jac}(X) \simeq (\mathbb{C}^\times)^N / \mathbb{C}^\times,$$

where $D_k(t) = (v_k)_0 \in S_{k-1, N-k}$.

Proposition 2.8. Let $1 \leq k, l \leq N$. Suppose that there exists two general divisors $D_k(t) \in S_{k-1, N-k}$, $D_l(t) \in S_{l-1, N-l}$ which satisfy the equation

$$\phi_{k-1, N-k}(D_k(t)) = \phi_{l-1, N-l}(D_l(t)) = \overline{F}(t).$$

Then, there exists a constant $C_{k,l}(t)$ such that

$$C_{k,l}(t) \cdot \frac{v_k(p)}{v_l(p)} = \frac{\Theta_k(p; F(t))}{\Theta_l(p; F(t))}, \quad p \in X \setminus \{P_1, \dots, P_N\}.$$

Proof. Since $(v_k(p)) = D_k(t) - (k-1)\infty_- - (N-k)\infty_+$, the function $\frac{v_k(p)}{v_l(p)}$ is a single-valued function over X whose divisor is $D_k(t) - D_l(t) + (k-l)(\infty_+ - \infty_-)$. On the other hand, by Corollary 2.7, the function $\frac{\Theta_k(p; F(t))}{\Theta_l(p; F(t))}$ possesses the same properties as well. Therefore, they must coincide as rational functions over X up to constant. \square

Lemma 2.9. For $p \in X$, let $x = x(p)$ be the rational coordinate of X_- and $y = y(p)$ be that of X_+ . The behaviors of $\frac{\Theta_k(p; F(t))}{\Theta_l(p; F(t))}$ when $p \rightarrow \infty_\pm$ are given by the followings:

$$\frac{\Theta_k(p; F(t))}{\Theta_l(p; F(t))} \underset{p \rightarrow \infty_-}{\sim} \frac{\tau_{k-1}x^{k-1} - \tau'_{k-1}x^{k-2} + \dots}{\tau_{l-1}x^{l-1} - \tau'_{l-1}x^{l-2} + \dots}, \quad \frac{\Theta_k(p; F(t))}{\Theta_l(p; F(t))} \underset{p \rightarrow \infty_+}{\sim} \frac{\tau_k y^{N-k} + \dots}{\tau_l y^{N-l} + \dots},$$

where $\boldsymbol{\lambda}^n := (\lambda_1^n, \dots, \lambda_N^n)^T$, $\mathbf{F} := (\exp(t\lambda_1), \dots, \exp(t\lambda_N))^T$ and

$$\tau_n := \tau_n(t) = (-1)^{n-1} \cdot \det(\mathbf{1}, \boldsymbol{\lambda}, \dots, \boldsymbol{\lambda}^{N-n+1}, \mathbf{F}, \mathbf{F}\boldsymbol{\lambda}, \dots, \mathbf{F}\boldsymbol{\lambda}^{n-1}), \quad (12)$$

$$\tau'_n := \tau'_n(t) = (-1)^{n-1} \cdot \det(\mathbf{1}, \boldsymbol{\lambda}, \dots, \boldsymbol{\lambda}^{N-n+1}, \mathbf{F}, \mathbf{F}\boldsymbol{\lambda}, \dots, \mathbf{F}\boldsymbol{\lambda}^{n-2}, \mathbf{F}\boldsymbol{\lambda}^n). \quad (13)$$

Proof. The formulas are proved directly from (11). \square

Corollary 2.10. We have $C_{k,l}(t) = \frac{\tau_k}{\tau_l}$, hence we conclude $\frac{\tau_k}{\tau_l} \cdot \frac{v_k(p)}{v_l(p)} = \frac{\Theta_k(p; F(t))}{\Theta_l(p; F(t))}$, ($\forall p \in X$).

Proof. From $v_k(p) \sim x^{k-1}$ ($p \rightarrow \infty_-$), Proposition 2.8 and Corollary 2.9. \square

Let $l < k$. When $p \rightarrow \infty_+$, it follows that $\frac{v_k(p)}{v_l(p)} \sim b_l b_{l+1} \cdots b_{k-1} y^{k-l}$. Therefore, from Proposition 2.8, we obtain the equation

$$b_l b_{l+1} \cdots b_{k-1} = C_{k,l}(t)^{-1} \frac{\tau_k}{\tau_l} = \frac{\tau_{l-1} \tau_k}{\tau_{k-1} \tau_l}.$$

By substituting $l = n$, $k = n+1$, we have

$$b_n = \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2}. \quad (14)$$

a_i is also given by the theta functions. From (5), we have $b_{n-1} v_{n-1}(p) + (a_n - \lambda) v_n(p) + v_{n+1}(p) = 0$. Then, $a_n = \lambda - b_{n-1} \frac{v_{n-1}(p)}{v_n(p)} - \frac{v_{n+1}(p)}{v_n(p)}$. By taking the limit $p \rightarrow \infty_-$ (see Lemma 2.9), we obtain

$$x - b_{n-1} \frac{\tau_{n-1}}{\tau_{n-2}} \frac{\Theta_{n-1}(p)}{\Theta_n(p)} - \frac{\tau_{n-1}}{\tau_n} \frac{\Theta_{n+1}(p)}{\Theta_n(p)} \sim x - \frac{\tau_{n-1}}{\tau_n} \frac{\tau_n x^n - \tau'_n x^{n-1} + \cdots}{\tau_{n-1} x^{n-1} - \tau'_{n-1} x^{n-2} + \cdots} \xrightarrow{x \rightarrow \infty} \frac{\tau'_n}{\tau_n} - \frac{\tau'_{n-1}}{\tau_{n-1}},$$

which implies

$$a_n = \frac{\tau'_n}{\tau_n} - \frac{\tau'_{n-1}}{\tau_{n-1}}. \quad (15)$$

Theorem 2.11. If the divisors D_n are general for $1 \leq n < N$, the variables $a_n = a_n(t)$, $b_n = b_n(t)$ in the Toda equation are given by (14) and (15). \square

In terms of the rational map

$$\Phi : \mathcal{T}_\Lambda \rightarrow \text{Jac}(X); \quad L \mapsto \overline{F}(t) = F(t) \bmod \mathbb{C}^\times, \quad (16)$$

which is defined from (6), Theorem 2.11 is expressed as follows:

Corollary 2.12. There exists a Zariski open set $\mathcal{U} \subset \text{Jac}(X)$ and a rational map $\Psi : \mathcal{U} \rightarrow \mathcal{T}_\Lambda; \overline{F}(t) \mapsto \{a_n(t), b_n(t)\}$ such that $\Psi \circ \Phi = \text{id}_{\mathcal{T}_\Lambda}$ and $\Phi \circ \Psi = \text{id}_{\mathcal{U}}$. Especially, Φ is injective. \square

3 Totally non-negative part of the phase space

In this section, we introduce some geometric characterization of the totally non-negative part (TNN part) of the phase space.

Denote $\mathcal{T}_\Lambda = H^{-1}(\gamma(\Lambda))$, the isospectral set associated with the spectrum set Λ (see §1). Let X be a singular curve defined in the previous section. In this section, we study the TNN part $\mathcal{T}_\Lambda^\geq \subset \mathcal{T}_\Lambda$ of the isospectral set. One of the main results of this paper is the characterization of $\Phi(\mathcal{T}_\Lambda^\geq)$ as a subset of $\text{Jac}(X)$.

3.1 Properties of totally non-negative tridiagonal matrices

We briefly introduce some of fundamental facts about TNN matrices. For readers who are interested in the topic of TNN matrices, we recommend the text [9]. We follow the notation of this text here.

Definition 3.1. A TNN matrix L is said to be *irreducible* if L^k is TP for some natural number k .

Theorem 3.2 (Gantmacher-Krein). All the eigenvalues of an irreducible TNN matrix are simple and positive.

Proof. See [9, Section 5], for example. \square

Definition 3.3. For an $N \times N$ matrix L and series of integers $1 \leq i_1 < i_2 < \dots < i_k \leq N$, $1 \leq j_1 < j_2 < \dots < j_k \leq N$, we denote by $L \begin{bmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{bmatrix}$ the $k \times k$ submatrix of L which consisted of i_1, i_2, \dots, i_k -th rows and j_1, j_2, \dots, j_k -th columns.

Proposition 3.4. ([9, Theorem 4.3]). An $N \times N$ tridiagonal matrix L is TNN if and only if both of the following (i) and (ii) are satisfied: (i) For any $1 \leq n \leq m \leq N$, $\det L \begin{bmatrix} n & n+1 & \dots & m \\ n & n+1 & \dots & m \end{bmatrix} \geq 0$. (ii) Each off-diagonal entry (*i.e.*, an entry which is not a diagonal entry) of L is nonnegative. \square

Corollary 3.5. An $N \times N$ tridiagonal matrix L is TNN if and only if all of the following (i)–(iii) are satisfied: (i) $\det L \geq 0$, (ii) $L \begin{bmatrix} 2 & 3 & \dots & N \\ 2 & 3 & \dots & N \end{bmatrix}$ is TNN, (iii) Each off-diagonal component of L is nonnegative.

Proof. Necessity is obvious. We prove the sufficiency. Suppose that L satisfies (i)–(iii). From Proposition 3.4, it suffices to prove $\det L \begin{bmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{bmatrix}$ is TNN for any $k = 1, 2, \dots, N$. For $k = N$, this is obvious from (i). Assume $1 \leq k \leq N - 1$. By Sylvester's relation, we have

$$\begin{aligned} & \det L \begin{bmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{bmatrix} \cdot \det L \begin{bmatrix} 2 & 3 & \dots & k+1 \\ 2 & 3 & \dots & k+1 \end{bmatrix} \\ &= \det L \begin{bmatrix} 1 & 2 & \dots & k+1 \\ 1 & 2 & \dots & k+1 \end{bmatrix} \cdot \det L \begin{bmatrix} 2 & 3 & \dots & k-1 \\ 2 & 3 & \dots & k-1 \end{bmatrix} + \det L \begin{bmatrix} 1 & 2 & \dots & k \\ 2 & 3 & \dots & k+1 \end{bmatrix} \cdot \det L \begin{bmatrix} 2 & 3 & \dots & k+1 \\ 1 & 2 & \dots & k \end{bmatrix}. \end{aligned}$$

From (ii), it follows that $\det L \begin{bmatrix} 2 & 3 & \dots & k-1 \\ 2 & 3 & \dots & k-1 \end{bmatrix} \geq 0$, $\det L \begin{bmatrix} 2 & 3 & \dots & k+1 \\ 2 & 3 & \dots & k+1 \end{bmatrix} \geq 0$. Since L is tridiagonal and (iii), we have $\det L \begin{bmatrix} 1 & 2 & \dots & k \\ 2 & 3 & \dots & k+1 \end{bmatrix} \geq 0$, $\det L \begin{bmatrix} 2 & 3 & \dots & k+1 \\ 1 & 2 & \dots & k \end{bmatrix} \geq 0$. Finally, we conclude that $\det L \begin{bmatrix} 1 & 2 & \dots & k+1 \\ 1 & 2 & \dots & k+1 \end{bmatrix} \geq 0$ implies $\det L \begin{bmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{bmatrix} \geq 0$. Therefore, by (i), we conclude that $\det L \begin{bmatrix} 1 & 2 & \dots & k \\ 1 & 2 & \dots & k \end{bmatrix}$ is TNN for any k . \square

Remark 3.6. Corollary 3.5 is also true if we replace the condition (ii) with the following (ii)':

$$(ii)' \quad L \begin{bmatrix} 1 & 2 & \dots & N-1 \\ 1 & 2 & \dots & N-1 \end{bmatrix} \text{ is TNN.}$$

Theorem 3.7. Let $N \geq 2$. For $N \times N$ irreducible TNN matrix L , set $Q := L \begin{bmatrix} 2 & 3 & \cdots & N \\ 2 & 3 & \cdots & N \end{bmatrix}$, $Q' := L \begin{bmatrix} 1 & 2 & \cdots & N-1 \\ 1 & 2 & \cdots & N-1 \end{bmatrix}$. Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$ be the spectrum of L , $0 < \mu_1 < \mu_2 < \cdots < \mu_{N-1}$ be the spectrum of Q and $0 < \mu'_1 < \mu'_2 < \cdots < \mu'_{N-1}$ be the spectrum of Q' . Then,

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 < \cdots < \mu_{N-1} < \lambda_N, \quad (17)$$

$$0 < \lambda_1 < \mu'_1 < \lambda_2 < \mu'_2 < \lambda_3 < \cdots < \mu'_{N-1} < \lambda_N. \quad (17')$$

Proof. [9, Proposition 5.4]. \square

Proposition 3.8. Assume $N \geq 2$. Suppose that L is a tridiagonal matrix whose off-diagonal entries are positive. Let Q, Q' be the same matrices as in Theorem 3.7, $\lambda_1 < \cdots < \lambda_N$ the spectrum of L , μ_1, \dots, μ_{N-1} the spectrum of Q and $\mu'_1, \dots, \mu'_{N-1}$ the spectrum of Q' . Then, the followings are equivalent: (i) L is TNN, (ii) (17) holds, (iii) (17') holds.

Proof. We prove by induction on $N \geq 2$. The case of $N = 2$ is directly proven. Assume $N \geq 3$ and λ_i, μ_i, μ'_i to satisfy (17). Let $L_\lambda := L - \lambda E$, $X(\lambda) := \det L_\lambda$, $X_1(\lambda) := \det L_\lambda \begin{bmatrix} 2 & 3 & \cdots & N \\ 2 & 3 & \cdots & N \end{bmatrix}$, $X_N(\lambda) := \det L_\lambda \begin{bmatrix} 1 & 2 & \cdots & N-1 \\ 1 & 2 & \cdots & N-1 \end{bmatrix}$, $X_{1,N}(\lambda) := \det L_\lambda \begin{bmatrix} 2 & 3 & \cdots & N-1 \\ 2 & 3 & \cdots & N-1 \end{bmatrix}$, $Y(\lambda) := \det L_\lambda \begin{bmatrix} 2 & 3 & \cdots & N \\ 1 & 2 & \cdots & N-1 \end{bmatrix}$ and $Z(\lambda) := \det L_\lambda \begin{bmatrix} 1 & 2 & \cdots & N-1 \\ 2 & 3 & \cdots & N \end{bmatrix}$. From Sylvester's relation (see [9, (p.5, Eq. (1.2))], for example), we have the equation

$$X(\lambda)X_{1,N}(\lambda) = X_1(\lambda)X_N(\lambda) - Y(\lambda)Z(\lambda). \quad (18)$$

Since L is tridiagonal and its off-diagonal components are positive, $Y(\lambda)$ and $Z(\lambda)$ are positive constants. Then, $X(\lambda)X_{1,N}(\lambda) < X_1(\lambda)X_N(\lambda)$. Substituting $\lambda = \mu_i$ ($i = 1, \dots, N-1$), we obtain $X(\mu_i)X_{1,N}(\mu_i) < 0$. From (17), the inequalities

$$X(0) = \det L > 0, \quad X(\mu_1) < 0, \quad X(\mu_2) > 0, \quad X(\mu_3) < 0, \dots$$

hold, which imply

$$X_{1,N}(\mu_1) > 0, \quad X_{1,N}(\mu_2) < 0, \quad X_{1,N}(\mu_3) > 0, \dots$$

By the intermediate value theorem, there exists some real number $\mu_i < c_i < \mu_{i+1}$, ($i = 1, \dots, N-2$) with $X_{1,N}(c_i) = 0$. As $X_{1,N}(\lambda)$ is a polynomial in λ of degree $(N-2)$, all the roots of $X_{1,N}(\lambda)$ are expressed as $\lambda = c_1, \dots, c_{N-2}$. By seeing the $(N-2) \times (N-2)$ matrix $L \begin{bmatrix} 2 & 3 & \cdots & N-1 \\ 2 & 3 & \cdots & N-1 \end{bmatrix}$ as a submatrix of Q , we conclude that Q is TNN by hypothesis of induction. From Corollary 3.5, also L is TNN. \square

3.2 Characterization of the TNN part

From Theorem 3.2, $\mathcal{T}_\Lambda^\geq \neq \emptyset$ implies that Λ consists of distinct N positive numbers. Hereafter, we assume $\Lambda = \{\lambda_1, \dots, \lambda_N\}$ ($0 < \lambda_1 < \cdots < \lambda_N$). Let X be the singular curve defined in the previous section and $\text{Jac}(X) = (\mathbb{C}^\times)^N / \mathbb{C}^\times$ be the generalized Jacobi variety of X .

The following theorem gives the characterization of the TNN part of the Toda flow.

Theorem 3.9. The image $\Phi(\mathcal{T}_\Lambda^\geq) \subset \text{Jac}(X)$ of the TNN part is expressed as

$$\Phi(\mathcal{T}_\Lambda^\geq) = \{[F_1 : F_2 : \cdots : F_N] \in \text{Jac}(X) \mid (-1)^i F_i > 0 \text{ for all } i\}.$$

Proof. Set $M = \{[F_1 : F_2 : \dots : F_N] \in \text{Jac}(X) \mid (-1)^i F_i > 0 \text{ for all } i\}$.

(Proof of $\Phi(\mathcal{T}_\Lambda^\geq) \subset M$.) Suppose $L \in \mathcal{T}_\Lambda^\geq$. From the construction of Φ , we have $\Phi(L) = [X_1(\lambda_1) : X_1(\lambda_2) : \dots : X_1(\lambda_N)]$, where $X_1(\lambda) = \det L_\lambda \begin{bmatrix} 2 & 3 & \dots & N \\ 2 & 3 & \dots & N \end{bmatrix}$. By Theorem 3.7, we have $(-1)^{i-1} X_1(\lambda_i) > 0$ for all i which implies $\Phi(L) \in M$.

(Proof of $\Phi(\mathcal{T}_\Lambda^\geq) \supset M$.) Let $[F_1 : \dots : F_N] \in M$. Without loss of generality, we assume $(-1)^{i-1} F_i > 0$ for all i . Define τ_n and τ'_n by equations (12), (13) and set

$$b_n := \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \quad a_n := \frac{\tau'_n}{\tau_n} - \frac{\tau'_{n-1}}{\tau_{n-1}}.$$

Let L be the Lax matrix (§2.1) defined by $\{a_n, b_n\}$. Then it follows that $\Phi(L) = [F_1 : \dots : F_N]$. Let us prove $L \in \mathcal{T}_\Lambda^\geq$. For a series of natural numbers $1 \leq i_1 < i_2 < \dots < i_n \leq N$, set $\Delta_{i_1, i_2, \dots, i_n} := \prod_{1 \leq a < b \leq n} (\lambda_{i_b} - \lambda_{i_a})$. Denote

$$\Delta_{i_1, i_2, \dots, i_n}^\dagger := \Delta_{j_1, j_2, \dots, j_{N-n}},$$

where $\{1, 2, \dots, N\} = \{i_1, i_2, \dots, i_n\} \sqcup \{j_1, j_2, \dots, j_{N-n}\}$. By the Laplace expansion, τ_n is expanded as follows:

$$\tau_n = \sum_{1 \leq i_1 < i_2 < \dots < i_{n-1} \leq N} (-1)^{i_1 + i_2 + \dots + i_{n-1}} \cdot F_{i_1} F_{i_2} \dots F_{i_{n-1}} \cdot \Delta_{i_1, \dots, i_{n-1}} \cdot \Delta_{i_1, \dots, i_{n-1}}^\dagger.$$

By the inequalities $0 \leq \lambda_1 < \dots < \lambda_N$ and $(-1)^{i-1} F_i > 0$, all the numbers $\Delta_{i_1, \dots, i_{n-1}}$, $\Delta_{i_1, \dots, i_{n-1}}^\dagger$, τ_n , b_n must be positive. Since b_n is a ratio of τ_n 's, the Lax matrix L is a matrix whose off-diagonal components are positive. Let $X_1(\lambda)$ be the $(1, 1)$ -cofactor of the matrix $L - \lambda E$. By definition of the Abel-Jacobi map, there exists some complex number c such that $X_1(\lambda_i) = c \cdot F_i$, ($i = 1, \dots, N$). Since $X_1(\lambda)$ is a real polynomial of degree $(N-1)$ whose leading coefficient is $(-1)^{N-1}$, c must be real and positive. Hence, $(-1)^{i-1} X_1(\lambda_i) > 0$ ($\forall i$). By the intermediate value theorem, there exists a real number $\lambda_i < \mu_i < \lambda_{i+1}$ with $X_1(\mu_i) = 0$. By applying Proposition 3.8 to L , the tridiagonal matrix whose off-diagonal components are positive, we conclude that L is TNN. Then, $L \in \mathcal{T}_\Lambda^\geq$. \square

3.3 The real structure

Let $\mathcal{T}_\Lambda^\mathbb{R} := \mathcal{T}_\Lambda \cap M_N(\mathbb{R})$ be the real part of \mathcal{T} and

$$\text{Jac}(X)_\mathbb{R} := \{[F_1 : \dots : F_N] \mid F_i \in \mathbb{R} \setminus \{0\}\}$$

be that of $\text{Jac}(X)$. Since the construction of Φ is closed in the reals, the linearization map $\Phi : \mathcal{T}_\Lambda \rightarrow \text{Jac}(X)$ induces the algebraic map

$$\Phi_\mathbb{R} : \mathcal{T}_\Lambda^\mathbb{R} \rightarrow \text{Jac}(X)_\mathbb{R}$$

of real algebraic varieties.

The real part $\text{Jac}(X)_\mathbb{R}$ consists of 2^{N-1} connected components of dimension $N-1$. Let $\text{Jac}(X)_\mathbb{R}^0 \subset \text{Jac}(X)_\mathbb{R}$ be the connected component which contains the identity element $[1 : 1 : \dots : 1]$. Theorem 3.9 states the existence of the isomorphism

$$\Phi_\geq : \mathcal{T}_\Lambda^\geq \rightarrow \text{Jac}(X)_\mathbb{R}^0$$

of two semi-algebraic varieties.

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